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A. Berezovski
Tallinn University of Technology, arkadi.berezovski@cs.ioc.ee

Juri Engelbrecht
Tallinn University of Technology, je@ioc.ee

Mihhail Berezovski
Tallinn University of Technology, berezovm@erau.edu

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Dispersive Wave Equations for Solids with Microstructure

Arkadi Berezovski, Jüri Engelbrecht and Mihhail Berezovski

Tallinn University of Technology, Estonia, e-mail: Arkadi.Berezovski@cs.ioc.ee

Abstract The dispersive wave motion in solids with microstructure is considered in the one-dimensional setting in order to understand better the mechanism of dispersion. It is shown that the variety of dispersive wave propagation models derived by homogenization, continualization, and generalization of continuum mechanics can be unified in the framework of dual internal variables theory.

Key words: Dispersive wave, microstructured solids, internal variables

1 Introduction

Several modifications of wave equation are proposed to describe wave propagation in heterogeneous materials reflecting dispersion effects, such as the linear version of the Boussinesq equation for elastic crystals [1, 2, 3, 4, 5], the Love-Rayleigh equation for rods accounting for lateral inertia [6, 7, 8, 9, 10], the Maxwell-Rayleigh equation of anomalous dispersion [1], the causal model for the dispersive wave propagation [11], and the Mindlin-type model [12].

All the equations listed above are based either on homogenization [2, 3, 9], or on continualisation [4, 8, 11], or on generalized continuum theories [5, 12]. There is a clear need in understanding their structure from a unified viewpoint. In what follows, the description of the non-dissipative dispersive wave propagation is unified by the dual internal variable approach [13].

2 Thermomechanics in one dimension

In the case of thermoelastic conductors of heat, the one-dimensional motion is governed by local balance laws for linear momentum and energy (no body forces)

$$(\rho v)_t - \sigma_x = 0, \tag{1}$$

$$\left(\frac{\rho v^2}{2} + E\right)_t - (\sigma v - Q)_x = 0, \tag{2}$$

and by the second law of thermodynamics

$$S_t + \left(\frac{Q}{\theta} + K\right)_x \geq 0. \tag{3}$$

Here t is time, subscripts denote derivatives with respect time and space, respectively, ρ is the matter density, $v = u_t$ is the physical velocity, σ is the Cauchy stress, E is the internal energy per unit volume, S is the entropy per unit volume, θ is temperature, Q is the material heat flux, the "extra entropy flux" K vanishes in most cases, but this is not a basic requirement.

3 Internal variables

In the framework of the phenomenological continuum theory it is assumed that the influence of the microstructure on the overall macroscopic behavior can be taken into account by the introduction of an internal variable φ which is associated with the distributed effect of the microstructure. In the dual internal variable theory [13], an auxiliary internal variable ψ is used additionally. Then the free energy W is specified as the general sufficiently regular function of the strain, temperature, internal variables φ, ψ and their space derivatives

$$W = \overline{W}(u_x, \theta, \varphi, \varphi_x, \psi, \psi_x). \tag{4}$$

The corresponding equations of state are given by

$$\begin{gathered}\sigma := \frac{\partial \overline{W}}{\partial u_x}, \quad S := -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \\ \xi := -\frac{\partial \overline{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \psi_x}.\end{gathered} \tag{5}$$

The dissipation inequality (3) can be rewritten as

$$(\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t - (Q/\theta + K)\theta_x + (\eta\varphi_t + \zeta\psi_t + \theta K)_x \geq 0. \tag{6}$$

Following [14], we chose the non-zero extra entropy flux K in the form

$$K = -\theta^{-1}\eta\varphi_{\,t} - \theta^{-1}\zeta\psi_{\,t}. \tag{7}$$

Such a choice allows us to reduce the dissipation inequality (6) to

$$(\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t - \left(\frac{Q - \eta\varphi_t - \zeta\psi_t}{\theta}\right)\theta_x \geq 0. \tag{8}$$

In this case, the dissipation is clearly decomposed into intrinsic and thermal parts. The latter means that the dissipation inequality in the isothermal case reduces to

$$(\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t \geq 0. \tag{9}$$

In the case of zero dissipation, Eq. (9) yields that the evolution equations for internal variables can be represented in the form [13]

$$\varphi_t = R(\xi - \zeta_x), \qquad \psi_t = -R(\tau - \eta_x), \tag{10}$$

where R is an arbitrary coefficient.

4 Constitutive model

Having the evolution equations for internal variables in the non-dissipative case, we can derive a microstructure model. We start with the free energy dependence in the form

$$\overline{W} = \frac{\rho c^2}{2} u_x^2 + A u_x \varphi + \tilde{A} u_x \varphi_x + a u_x \left(\frac{dF(u)}{du}\right)_x + \frac{1}{2} B\varphi^2 + \frac{1}{2} C\varphi_x^2 + \frac{1}{2} D\psi^2, \tag{11}$$

where c is the elastic wave speed, $A, \tilde{A}, B, C$, and D are material parameters, $F(u)$ is the nonlinear contribution at macroscale, a is a scaling coefficient. For simplicity, we include only the contribution of the second internal variable itself. In this case, stresses are calculated as follows:

$$\begin{aligned} \sigma &= \frac{\partial \overline{W}}{\partial u_x} = \rho c^2 u_x + A\varphi + \tilde{A}\varphi_x + a\left(\frac{dF(u)}{du}\right)_x, \\ \eta &= -\frac{\partial \overline{W}}{\partial \varphi_x} = -\tilde{A} u_x - C\varphi_x, \quad \zeta = -\frac{\partial \overline{W}}{\partial \psi_x} = 0. \end{aligned} \tag{12}$$

The interactive internal forces τ and ξ are, respectively,

$$\tau = -\frac{\partial \overline{W}}{\partial \varphi} = -A u_x - B\varphi, \qquad \xi = -\frac{\partial \overline{W}}{\partial \psi} = -D\psi. \tag{13}$$

The evolution equations (10) in the case of zero dissipation take the form

$$\varphi_t = R(\xi - \zeta_x) = -RD\psi, \tag{14}$$

$$\psi_t = -R(\tau - \eta_x) = R(A u_x + B\varphi - \tilde{A} u_{xx} - C\varphi_{xx}). \tag{15}$$

It follows immediately from Eqs. (14), (15) that the evolution equation for the primary internal variable (14) can be rewritten as the hyperbolic equation

$$\varphi_{tt} = R^2 D(\tau - \eta_x). \tag{16}$$

Accordingly, the balance of linear momentum (1) results in

$$\rho u_{tt} = \rho c^2 u_{xx} + A\varphi_x + \tilde{A}\varphi_{xx} + a[F'(u)]_{xx}, \tag{17}$$

and the evolution equation for the primary internal variable (16) gives

$$I\varphi_{tt} = C\varphi_{xx} + \tilde{A}u_{xx} - Au_x - B\varphi, \tag{18}$$

where $I = 1/(R^2 D)$ is an internal inertia measure.

Single dispersive wave equation. To derive the single equation we make following steps. We determine the first derivative of the internal variable from Eq. (18)

$$B\varphi_x = -I\varphi_{ttx} + C\varphi_{xxx} + \tilde{A}u_{xxx} - Au_{xx}. \tag{19}$$

The third mixed derivative φ_{ttx} follows from Eq. (17)

$$A\varphi_{ttx} = \left(\rho u_{tt} - \rho c^2 u_{xx} - a[F'(u)]_{xx}\right)_{tt} - \tilde{A}\varphi_{ttxx}. \tag{20}$$

The appeared fourth-order mixed derivative the internal variable is calculated by means Eq. (18)

$$I\varphi_{ttxx} = C\varphi_{xxxx} + \tilde{A}u_{xxxx} - Au_{xxx} - B\varphi_{xx}, \tag{21}$$

and, in its turn, the fourth-order space derivative is determined again from Eq. (17)

$$\tilde{A}\varphi_{xxxx} = \left(\rho u_{tt} - \rho c^2 u_{xx} - a[F'(u)]_{xx}\right)_{xx} - A\varphi_{xxx}. \tag{22}$$

Collecting all the results (19) - (22) and substituting them into the balance of linear momentum (17) we arrive at the dispersive wave equation

$$\begin{aligned} \rho u_{tt} - \rho c^2 u_{xx} - a[F'(u)]_{xx} &= \frac{C}{B}\left(\rho u_{tt} - \rho c^2 u_{xx} - a[F'(u)]_{xx}\right)_{xx} \\ - \frac{I}{B}\left(\rho u_{tt} - \rho c^2 u_{xx} - a[F'(u)]_{xx}\right)_{tt} &+ \frac{\tilde{A}^2}{B}u_{xxxx} - \frac{A^2}{B}u_{xx}. \end{aligned} \tag{23}$$

5 Examples of dispersive wave equations

5.1 Linear dispersive wave equations

Mindlin-type model. The Mindlin-type model [12] corresponds to $a = 0$ (no nonlinearity) and $\tilde{A} = 0$ (no coupling between gradients) in Eq. (23):

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} - \frac{A^2}{\rho B}u_{xx}. \tag{24}$$

The Maxwell-Rayleigh model of anomalous dispersion [1] corresponds to a special case of the latter equation with $C = 0$.

Causal model. Keeping the absence of nonlinearity in Eq. (23) and assuming $A = 0$ (no coupling between strain and internal variable; only gradients are coupled), we arrive at the causal model [11]:

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} + \frac{A'^2}{\rho B} u_{xxxx}. \tag{25}$$

The higher-order dispersive wave equations (24) and (25) differ from each other only by the last term in the right hand side. However, this difference is essential, because the second-order space derivative in Eq. (24) exhibits the slowing down the velocity of propagation, whereas the fourth-order derivative in Eq. (25) does not. At the same time, derivatives of the wave operator in Eq. (24) cannot be rearranged, whereas it is possible in Eq. (25) due to the additional fourth-order space derivative.

Unified model. The unified model includes both couplings mentioned above

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} + \frac{A'^2}{\rho B} u_{xxxx} - \frac{A^2}{\rho B} u_{xx}, \tag{26}$$

which generalizes both approaches [15].

5.2 Nonlinear dispersive wave: Boussinesq equation

The obtained dispersive wave equation (23) can be reduced to the Boussinesq equation under following assumptions:

1. $I = 0$, which means zero microinertia;
2. $G = 0$ that corresponds to the absence of nonlinearity in microstructure;
3. $A = 0$ (no coupling between strain and internal variable; only gradients are coupled);

As a result, Eq. (23) reduces to

$$\rho u_{tt} - \rho c^2 u_{xx} - a[F'(u)]_{xx} = \frac{C}{B}\left(\rho u_{tt} - a[F'(u)]_{xx}\right)_{xx} + \frac{\tilde{A}^2}{B} u_{xxxx}. \tag{27}$$

This equation belongs to the class of the dispersive wave equations which are characterized by the so-called "Boussinesq paradigm" [16], which means: i) bidirectionality of waves; ii) nonlinearity (of any order); iii) dispersion (of any order modelled by space and time derivatives of the fourth order at least).

This paradigm has its roots in the classical Boussinesq equation for waves in shallow water, to which Eq. (27) can be reduced by the choice of the nonlinearity function $F(u) = u^3$ [16] and $C = 0$

$$u_{tt} - c^2 u_{xx} = \left(\frac{3au^2}{\rho} + \frac{\tilde{A}^2}{\rho B} u_{xx} \right)_{xx}. \tag{28}$$

6 Conclusions

As it was shown on the example of one-dimensional wave propagation, nonlinear terms can be easily introduced in the framework of the dual internal variables approach resulting in a generalized nonlinear dispersive wave equation. A cubic macroscopic nonlinearity leads to the Boussinesq equation.

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